THE GIT CONSTRUCTION OF THE MODULI SPACE OF POINTED CURVES

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1. Introduction

This paper is an overview of the geometric invariant theory (GIT) construction of a compactification of $\mathrm{M}_{g,n}$, the moduli space of smooth, genus g curves with n marked points.

We begin with a general outline to describe the GIT approach of constructing compactifications of moduli spaces. We hope this section provides the reader with a big picture understanding of GIT and an appreciation for the ingredients involved. In Section 3 We then describe the GIT construction of the compactification of smooth genus g curves with no marked points, $\overline{M}_g$, as outlined in [1]. This will allow us to describe some building blocks and terms involved in compactifying $\mathrm{M}_{g,n}$. In Section 4.3 we give a description of the ingredients in the GIT compactification of $\mathrm{M}_{g,n}$ due to Swinarski [3]. We conclude in the final section with an example of determining stability of elliptic curves.

2. Brief Overview of GIT

Suppose we are given a set $\mathcal{M}$ of isomorphism classes of varieties. It would be desirable to somehow construct a space $\overline{\mathcal{M}}$ which contains $\mathcal{M}$ as a dense open subset and where the points in $\overline{\mathcal{M}}/\mathcal{M}$ are isomorphism classes of degenerations of the varieties representing classes of $\mathcal{M}$ and which appear in a predictable way (meaning these degenerate objects are what one obtains by considering continual variations from the objects in $\mathcal{M}$). In other language, we would like to construct a projective scheme $\overline{\mathcal{M}}$ that is a compactification of $\mathcal{M}$. This means, the space $\overline{\mathcal{M}}$ contains $\mathcal{M}$ as a dense open subset (so that it is a compactification) and that $\overline{\mathcal{M}}$ contains unique limits from $\mathcal{M}$ (so that it is a projective scheme). The goal of GIT is to produce such a compactification $\overline{\mathcal{M}}$.

Typically there are more properties on the scheme structure of $\overline{\mathcal{M}}$ we would like. GIT provides a rich method for constructing such objects with these nice properties for certain isomorphism classes of varieites. For our purposes, the above description is enough one might desire for providing an idea behind the methods involved in GIT.

The GIT approach of constructing the compactification $\overline{\mathcal{M}}$ is to construct such a space as a quotient. The quotient we construct is of some larger parameter space $\mathcal{K}$ by an action of a group G. To make this work, the space $\mathcal{K}$ is a larger, nice parameter space which paramertirizes the objects representing classes in $\mathcal{M}$ with some extra data. The group G acts on $\mathcal{K}$ in such a way so that the set of points of $\mathcal{K}$ parameterizing the extra data on

Date: December 4, 2015.

an element $\mathcal{C}$ of $\mathcal{M}$ forms a single G orbit in $\mathcal{K}$. When we take the quotient $\mathcal{K}/G$, the closed points of $\mathcal{K}/G$ correspond to closed points in $\mathcal{M}$.The steps involved in GIT include constructing the space $\mathcal{K}$ and describing the action of G on $\mathcal{K}$ so that the quotient is the desired compactification. We now describe the general idea of how one actually creates such a quotient.

2.1. **A simple idea for constructing a quotient.** Suppose we have a projective subvariety $\mathcal{K} \subset \mathbb{P}(W)$ where W is a linear representation of a reductive group G (i.e., there is an action of G on W that is diagonalizable) and we want to take a quotient of $\mathcal{K}$ by a group G. To create the quotient $\mathcal{K}/G$, we must create a quotient map $\mathcal{K} \to \mathcal{K}/G$. To do this, we define such a map locally and then glue these maps together.

If $X = Spec(R) \subset \mathcal{K}$ is a G-invariant affine subscheme of $\mathcal{K}$ and R^G is the subring of G-invariant elements of R, then we define the quotient X/G to be $Spec(R^G)$ and the rational map

$$\phi : X \dashrightarrow X/G$$

is dual to the inclusion $f : R^G \hookrightarrow R$. That is, if $\mathfrak{p} \in Spec(R^G)$ so that $\mathfrak{p}$ is a prime ideal of R, then $\phi(\mathfrak{p}) = f^{-1}(\mathfrak{p})$. We must patch such local maps together to form a quotient map on the projective scheme $\mathcal{K}$ to the quotient $\mathcal{K}/G$,

$$q : \mathcal{K} \dashrightarrow \mathcal{K}/G.$$

Here $\mathcal{K}/G$ is the projective space $\mathrm{Proj}(\mathbb{C}[\mathcal{K}]^G)$. In order to glue these locally defined maps together and create the desired larger quotient map, one must describe the action of G on an ample line bundle of $\mathcal{K}$ that behaves well on the fibers of the ample line bundle (the reason for this is elaborated on in Section 2.4). Such an action on an ample line bundle is called a *linearization.*

In summary, this procedure produces a map $q : \mathcal{K} \dashrightarrow \mathcal{K}/G$ with $\mathcal{K} = \mathrm{Proj}(\mathbb{C}[\mathcal{K}])$ and $\mathcal{K}/G = \mathrm{Proj}(\mathbb{C}[\mathcal{K}]^G)$ so that the quotient map is given by taking the values of homogeneous G-invariant polynomials on $\mathcal{K}$.

There is a lot of work required to make all of the definitions make sense in the above procedure. For example, the ring of invariants $\mathbb{C}[\mathcal{K}]^G$ must be suitably nice in order to define such a map into $\mathrm{Proj}(\mathbb{C}[\mathcal{K}]^G)$. For example, it must be Noetherian, it can't be too large, and it can't be too small. It turns out such questions are quite difficult to address and are related to "Hilbert's 14th Problem." For further details, see a the classical text on the subject [2]. We do not go into these issues here, rather we summarize and further describe the two main ingredients one must have in order to construct a quotient map as described above and give motivation for the importance of these ingredients.

2.2. **Ingredients for GIT.** There are two ingredients necessary to describe a space of good quotients $\mathcal{K}/G$ (i.e., a space $\mathcal{K}/G$ who's closed points are orbits of G in $\mathcal{K}$). They are as follows,

(1) $\mathcal{K}$, a parameter space whose objects parametrize those in $\mathcal{M}$ with more data, and
(2) a linearization of an ample line bundle on $\mathcal{K}$.

2.3. **The Parameter Space.** Again, we would like to construct a quotient $\mathcal{K}/G$ whose closed points correspond to G-orbits of $\mathcal{K}$. With a potential parameter space $\mathcal{K}$, we must restrict this space before taking a quotient to only those points in $\mathcal{K}$ whose G orbits are indeed closed. We call a point $x \in \mathcal{K}$ a *semistable* point if it has such a nice G-orbit. The semistable locus of $\mathcal{K}$, denoted by $\mathcal{K}^{ss}$, represents all such stable points. Results from Mumford have shown that when restricting to the semistable locus $\mathcal{K}^{ss}$, the quotient $\mathcal{K}^{ss}/G$ does indeed consists of closed points representing G-orbits and will be a compactification of the original space of isomorphism classes, $\mathcal{M}$.

In more GIT generality, there are three classifications of points in a parameter space $\mathcal{K}$ in relation to an action of G. This classification consists of points as *semistable*, *polystable*, and *stable*, [2]. Naively, *semisetable* points are those whose G orbits do not contain 0, *polystable* are points that are semistable and have a closed orbit, and *stable* are points that are polystable and the dimension of the orbit is equal to the dimension of G. The amazing insights of Mumford (from examples of Hilbert) show that the classification of all three types of points can be determined simply be considering how the torus of G acts on the point. Explicitly, Mumford's result is called "The Hilbert-Mumford Numerical Criterion." This criterion provides a simple condition for determining the classification of a given point $x \in \mathcal{K}$ in a numerical way. See [1, Theorem 4.17] for the full statement. We will see this numerical criteria explicitly for elliptic curves in Section 5.

2.4. **The Linearization.** Referring back to section 2.1 on the simple idea of GIT, recall that when constructing a locally defined quotient map, we must consider how we will glue the local maps together. To do this, we embed the space $\mathcal{K}$ in some projective space. This brings into the picture an ample line bundle $\mathcal{L}$ on $\mathcal{K}$. Indeed, any embedding provides an ample line bundle (i.e., by pulling back the the ample bundle $\mathcal{O}(1)$ on the target space). We have an action of G on $\mathcal{K}$ but to describe the action on $\mathcal{K}$ as an embedded variety, we must define an action of G on the line bundle $\mathcal{L}$. We would like such an action to exhibit some nice properties, particularly this action must fix the fibers of the line bundle. An action of G defined on $\mathcal{L}$ with this property is called a *linearization* of G.

In the case when the objects we are parametrizing are genus g cures with n marked points, we will see an example of such a linearization (see Section 4.2).

3. Summary of construction of $\overline{\mathrm{M}}_g$ using GIT

We now describe the ingredients in the GIT construction of the compactification of the moduli space M_g of smooth curves $\mathcal{C}$ of genus g. This will introduce some of the objects involved in constructing $\overline{M}_{g,n}$ as a GIT quotient in Section 4.

The first ingredient we need is the parameter space $\mathcal{K}$ of which we will take a quotient. As we described in the previous section, to do this, we consider a larger space of pairs consisting of curves with some extra data. With such a collection of pairs, we then want to take a quotient which will identify the additional data in such a way that the objects in our quotient are indeed the isomorphism classes making up M_g.

The larger space we consider consists of curves with an embedding into projective space by a power of their canonical line bundle, $\omega_{\mathcal{C}}$. The canonical line bundle is the line bundle associated to the canonical divisor class of the curve. Recall, for a curve $\mathcal{C}$, the canonical divisor class, $\mathbb{K}$, is the linear equivalence class consisting of divisors of meromorphic differentials on $\mathcal{C}$. Thus, the line bundle $\omega_{\mathcal{C}}$ has global sections consisting of everywhere differential forms on $\mathcal{C}$. One can check that for all g, the line bundle $\omega_{\mathcal{C}}$ is ample. As an example, the canonical line bundle of an elliptical curve is trivial. The property of $\omega_{\mathcal{C}}$ being ample means that some power of it provides us with an embedding of $\mathcal{C}$ into projective space.

It is stated in [1, p. 194] that for any $n \geq 3$ the n^{th} power of the canonical class, $\omega_{\mathcal{C}}^{\otimes n}$, embeds $\mathcal{C}$ into projective space of dimension $(2n-1)(g-1)-1$ as a degree $2(g-1)n$ curve. We call this embedding the "n^{th}-canonical embedding of $\mathcal{C}$." Specifically then, the parameter space $\mathcal{K}$ consists of pairs $(\mathcal{C}, \phi : \mathcal{C} \to \mathbb{P}^{(2n-1)(g-1)-1})$ where $\mathcal{C}$ is a curve of genus g and ϕ is the n^{th}-canonical embedding of $\mathcal{C}$ into $\mathbb{P}^{(2n-1)(g-1)-1}$.

This space conveniently lives in a larger scheme called the *Hilbert Scheme*. The Hilbert Scheme $\mathcal{H}_{P,r}$ parameterizes subschemes X of r-dimensional projective space, $X \subset \mathbb{P}^r$ with Hilbert polynomial P. Recall, for a subscheme $X \subset \mathbb{P}^r$, the *Hilbert polynomial* is defined as a polynomial that agrees with the Hilbert function for large values. Hence, for large m, the Hilbert polynomial computes the dimension of global sections of an ample line bundle. That is, $P_X(m) = h^0(X, \mathcal{O}_X(m))$ where $\mathcal{O}_X$ is an ample line bundle on X. If $deg(X) = d$ and $dim(X) = s$ then the polynomial $P_X(m)$ has leading term $\frac{dm^s}{s!}$. For more details on a construction of this scheme, see [1, Sect. 1.B].

Since the subschemes of $\mathbb{P}^r$ we are interested in parametrizing are curves (and so the dimension of the subschemes of interest are one) the Hilbert polynomial of any such subscheme is of the form $P(m) = dm - g + 1$, where d is the degree of the curve and g is the genus. Hence, the only variant we must consider is the degree and genus. We denote this Hilbert scheme consisting of curves of $\mathbb{P}^r$ with a fixed genus g and degree d as $\mathcal{H}_{d,g,r}$. One can either realize this space as parametrizing abstract curves of arithmetic genus g plus a very ample linear system of degree d or as parametrizing subcurves of $\mathbb{P}^r$ of degree d and genus g. The abstract setting allows us to describe a subsheme of $\mathcal{H}_{d,g,r}$ which is exactly the space of pairs we described in the previous paragraph. We now connect these two ideas.

By considering a space of genus g curves with an embedding into projective space by $\omega_C^{\otimes n}$, we find ourselves describing a subscheme of the Hilbert Scheme $\mathcal{H}_{d,g,r}$ for $d = 2(g-1)n$ and $r = (2n-1)(g-1)-1$. As we just described, abstractly $\mathcal{H}_{d,g,r}$ can be considered as the space parametrizing abstract curves of arithmetic genus g plus a very ample linear system of degree d. Explicitly then, let $\mathcal{K} \subset \mathcal{H}_{d,g,r}$ be the subset in $\mathcal{H}_{d,g,r}$ of curves with very ample linear system of degree d isomorphic to the n^{th}-canonical embedding of C, $\omega_C^{\otimes n}$.

4. Summary of construction of $\overline{\mathrm{M}}_{g,n}$ using GIT

In [3], Swinarski uses GIT to construct a compactification $\overline{M}_{g,\mathcal{A}}$ for the moduli space of weighted pointed curves. As a consequence of describing the parameter space and linearizations for when such weighted pointed curves are stable, he obtains a result on the stability of pointed curves [3, Theorem 7.1]. Taking a quotient of this parameter space with a specified linearization produces the compactification of interest [3, Theorem 7.2]. We describe the parameter space and linearization now. In this discussion, we have more data associated to the objects initially being parametrized. These objects are *weighted pointed stable curves*, $(C, P_1, ..., P_n, \mathcal{A})$ which consist of the following data:

- C, a reduced connected projective algebraic curve of genus g with at worst nodes as singularities,
- P_i, n distinct points that lie on C and are ordered,
- $\mathcal{A} = (a_1, ..., a_n)$, an n-tuple such that
  - $a_i \in \mathbb{Q} \cap [0, 1]$,
  - $a_i = 0$ if P_i is a node, and
  - the $\mathbb{Q}-$line bundle $\omega(\sum a_i P_i)$ is ample on C.

In the final condition, the line bundle $\omega(\sum a_i P_i)$ is simply the line bundle associated to the divisor class containing $K + a_1P_1 + ... + a_nP_n$ where K is a canonical divisor.

4.1. **The Parameter Spaces.** Similar to the parameter space used in the construction of $\overline{\mathrm{M}}_g$ (see Section 3), the parameter space here involves a Hilbert scheme. If $(C, P_1, ..., P_n, \mathcal{A})$ is a weighted marked curve of genus g, then $\omega(\sum a_i P_i)$ is ample and so a large enough multiple of this line bundle gives an embedding of C into some projective space $\mathbb{P}^r$. It is natural then to consider pairs in a Hilbert Scheme consisting of the curve C and this ample line bundle $\omega(\sum a_i P_i)$. Doing so again describes a subscheme $\mathcal{K}$ of a Hilbert Scheme $\mathcal{H}_{d,g,r}$. The subscheme of interest $\mathcal{K}$ parameterizes those varieties whose ample line bundle is isomorphic to $\omega(\sum a_i P_i)$.

The curves in our investigation also include a collection of n marked points. To record this information, we include the images of these n points on the embedding of C in $\mathbb{P}^r$ by a power of $\omega(\sum a_i P_i)$.

In final, the parameter space we consider is a subscheme $\mathcal{I}$ in the following product

$$\mathcal{H}_{d,g,r} \times \underbrace{\mathbb{P}^r \times ... \times \mathbb{P}^r}_{\text{n copies}}.$$

The subscheme $\mathcal{I}$ consists of elements $(K, P_1, ..., P_n)$ where K is in the subscheme $\mathcal{K}$ in a Hilbert scheme described in the previous paragraph, and $(P_1, ..., P_n) \in \Pi^n\mathbb{P}^r$ are the images of the n marked points of C in $\mathbb{P}^r$ embedded by a power of $\omega(\sum a_i P_i)$.

4.2. **The Linearizations.** As is stated in [3, p. 5], a linearization of a line bundle on $\mathcal{I} \subset \mathcal{H}_{d,g,r} \times \Pi^n\mathbb{P}^r$ is given by an embedding of $\mathcal{H}_{d,g,r} \times \Pi^n\mathbb{P}^r$ into a very large projective space. Such an embedding is by an n+1-tuple $(m, m_1, ..., m_n)$. Indeed, the m determines an embedding of the Hilbert scheme $\mathcal{H}_{d,g,r}$ into a Grassmannian and each m_i determines an m_i-uple embedding of $\mathbb{P}^r$ into another projective space. A Segre embedding of all of these projective spaces yields an embedding of the full $\mathcal{H}_{d,r,g} \times \Pi^n\mathbb{P}^r$ into a very, very large projective space.

Furthermore, to specify an embedding (and thus a linearization), it is only necessary to specify the ratio between m and each m_i. For m sufficiently large and γ depending on the data of the weighted marked curve, Swinarski uses the following linearization $(m_1, ..., m_n)$ on $\mathcal{I}$:

$$m_i := \gamma a_i m2. \tag{4.1}$$

4.3. **The final quotient.** Swinarski then claims in [3, Theorem 7.2] that for the subscheme $\mathcal{I}$ described above and for a sufficiently large m the linearization given by the m_i defined in 4.1, that when a quotient is take, we obtain the following

$$\mathcal{I}//SL(N+1) \cong \overline{M}_{g,\mathcal{A}}. \tag{4.2}$$

Furthermore, it is stated that for some small ϵ, if $1/2 + \epsilon < a_i < \frac{1}{2\gamma}$ for $i = 1, ..., n$, then

$$\mathcal{I}//SL(N+1) \cong \overline{M}_{g,n}, \tag{4.3}$$

obtaining a desired compactification of $\mathrm{M}_{g,n}$ using GIT.

Much of the work in the argument in [3] to show 4.2 is to describe the stability of a smooth pointed curve with respect to a one parameter subgroup. To do this, Swinarski begins with a 1-PS λ of $SL(N+1)$ and considers the action of λ on a smooth pointed curve. From this action, he obtains a way of partitioning into a sequence of subspaces the vector space $H^0(C, \mathcal{O}(1))$ of global section on our curve determined by an ample line bundle $\mathcal{O}(1)$. This is called a *filtration* of $H^0(C, \mathcal{O}(1))$. He modifies this filtration slightly so that he is able to translate the action of λ and determine stability of a smooth pointed curve with respect to a linearization.

5. An Example: GIT stability of elliptic curves

Much of the importance involved constructing quotients using GIT is to determine stability and semistability of the objects in a parameter space. To demonstrate this, we carry out an example of determining stability of elliptic curves. This example shows that when considering a space parametrizing all genus one curves, the stable ones are preciely those in the our original set of isomorphism classes, that is, smooth elliptic curves.

As we stated in Section 2.2, the first ingredient in constructing a GIT quotient is a parameter space $\mathcal{K}$. To construct a quotient whose orbits are identified with closed points in $\mathcal{M}$, the elements in $\mathcal{K}$ must be stable with respect to an action of a group G. In this section, we exam this deeply for the case of elliptic curves (i.e., genus one curve) and analyze the consequences of the stability condition for such curves. This discussion is from work in [1, Chap. IV].

Let $\mathcal{M}_1$ be the set of isomorphism classes of smooth genus one curves. The parameter space we consider is the space

$$\mathcal{K} = \mathbb{P}(Sym^3(\mathbb{C}^3)\check{}) = \mathbb{P}(\{x^3, x^2y, xy^2, x^2z, xz^2, xyz, y^3, y^2z, yz^2, z^3\}).$$

This is the projectivization of the space of degree three monomials in three variables. Recall, that any genus one curve can be expressed as the zero locus of a homogeneous degree three polynomial,

$$C = \mathbb{V}(f(x, y, z)).$$

Hence, any such polynomial (and so genus one curve) is naturally an element of the dual space $\mathcal{K}$ above determined by the coefficients of the monomials defining it.

Furthermore, consider the space

$$\mathcal{K} - \Delta$$

where Δ denotes the discriminant hypersurface (i.e. the collection of signal cubic curves). This space then parametrizes smooth genus one curves with a choice of homogeneous coordinates. Two genus one curves C and C' are isomorphic if there is an automorphism of $\mathbb{P}^2$ sending C to C'. In this way, the group $G = SL_3$ (the automorphism group of $\mathbb{P}^2$) acts on $\mathcal{K} - \Delta$. Hence, the space $\mathcal{K} - \Delta$ parametrizes genus one plane curves with a choice of homogeneous coordinates.

Let's then consider when a curve $C \in \mathcal{K}$ is stable with respect to this action. We saw in Section 2.3 that to show that C is stable, it is enough to show that C is stable with respect to every one parameter subgroups (1 PS), $\lambda : \mathbb{C}^* \to G$. This means, it is enough to consider the action of the torus on C.

For $G = SL_3$, a 1-PS, λ, has the following form:

$$\lambda(t) = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

where $a + b + c = 0$ (this is because the elements of SL_3 are 3×3 matrices with determinant one and the torus in SL_3 consist of the diagonal elements).

For a single monomial $x^i y^j z^k$, we see that $\lambda(t)$ has the following action:

$$x^i y^j z^k \mapsto t^{ai+bj+ck} x^i y^j z^k.$$

Let's then consider how such an element λ acts on a genus one curve, C. If C is given by the polynomial

$$f(x, y, z) = \sum_{i+j+k=3} c_{ijk} x^i y^j z^k,$$

then $\lambda(t)$ acts on C by action of the polynomial $f(x, y, z)$ the following way,

$$\lambda(t).f(x, y, z) \mapsto \sum_{i+j+k=3} c_{ijk} t^{ai+bj+ck} x^i y^j z^k.$$

With this understanding of the action of λ on C, we can interpret the Hilbert-Mumford Criteria to say the following:

$$C \text{ is } \lambda \text{ stable } \Leftrightarrow \text{ some } c_{ijk} \neq 0 \text{ for } ai + bj + ck < 0.$$

This means that when acting on C by $\lambda(t)$, there will be a term which appears in the new defining polynomial with a negative power of t.

One can analyze λ-stability in some specific cases and conclude (as is done in [1, p. 204]) that C is λ-stable if and only if C is smooth. By analyzing the symmetry involved in the

λ-stability condition, one can argue further, that λ-stability of a specific example is enough to conclude stability for any 1-PS. In conclusion, the following claim can be made:

$$C \text{ is stable } \Leftrightarrow C \text{ is smooth.}$$

This example shows that the parameter space considered when constructing a GIT quotient to represent isomorphism classes of smooth elliptic curves, does not pick up any more objects than was already considered. The only stable genus one curves are smooth curves. Furthermore, by analyzing semistability of genus one curves, we see that at worst only nodal curves will be included when expanding to the semistable locus.

This discussion then shows that the compacitificatoin $\overline{M}_1$ includes isomorphism classes of smooth curves and picks up isomorphism classes of curves with at worst nodal singularities.

References

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