NOTES ON TRIVIALIZATIONS OF VECTOR BUNDLE ON A CURVE WITH TRIVIAL DETERMINANT

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In this note we expand on the following result used in the isomorphism of generalized theta functions and conformal blocks (outlined in [Bea] and with further details in [Bea94]).

Proposition 0.1. *If E is a vector bundle over a curve, C, with trivial determinant and* $\mathrm{rk}(E) \geq 2$*, then E can be trivialized over $C - p$ for any $p \in C$.*

This explanation was first communicated to me by A. Gibney in an "Expanded Note" by P. Belkale and A. Gibney. Thank you also to R. Varley for helpful discussion in showing necessity of trivial determinant and other necessary editions.

In this explanation, we use the following facts:

Fact 0.2. *(Corollary II 3.2. Hartshorne) If D is a divisor on a curve C, then*

a) *If $deg(D) \geq 2g$ then $|D|$ has no base points. (i.e. the morphism obtained by D, ϕ_D is regular and particularly, $\mathcal{O}(D)$ will have global sections).*

b) *If $deg(D) \geq 2g+1$ then D is very amble (i.e., the morphism obtained by D is an embedding) and particularly, the line bundle $L(D)$ is generated by global sections.*

Fact 0.3. *Serre Duality and Riemann-Roch.*

a) *Serre Duality: For E is a vector bundle over a scheme X and K the canonical divisor of X, then $H^q(X,E) \cong H^{n-q}(X, K \otimes E^*)^*$*
b) *Line bundle L such that $deg(L) \geq 2g-1$ will have $h^1(L) = h^0(L^* \otimes K) = 0$.*
c) *Riemann Roch: For E a vector bundle over a curve C, we have $h^0(E) - h^1(E) = d + (1-g)\,\mathrm{rk}(E)$, where d is the degree of E.*
d) *For a line bundle L of degree $deg(L) \geq 2g-1$ and D_L a divisor associated to L, then $h^1(L) = h^0(L^* \otimes K) = h^0(K - D_L) = 0$.*
e) *Serre Vanishing: For $\mathcal{L}$ ample on X and $\mathcal{F}$ a coherent sheaf, then there is some $m \in \mathbb{N}$ such that $H^b(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for $b > 0$ and $n > m$.*
(Reference: http://math.stackexchange.com/questions/132751/riemann-roch-for-vector-bundles)

We now show a series of results to explain the Proposition above.

Proof. First, suppose E is a vector bundle over a curve C with trivial determinant.

1. There is a nonzero global section of $E \otimes \mathcal{O}(np)$ for any $p \in C$ and sufficiently large n.

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(i) For any point $p \in C$, np is a divisor on C. Hence, by Corollary II 3.2, we know that $h^0(\mathcal{O}(np)), h^0(\mathcal{O}(np-q)) \geq 0$ if both $deg(np), deg(np-q) \geq 2g+1$. Since $deg(np) = n$ and $deg(np-q) = n-1$, this will be the case if $n \geq 2g+1$. So we choose n such an $n \geq 2g+1$. This means $\mathcal{O}(p)$ is ample and so we can use Serre Vanishing.

(ii) We now apply Serre Vanishing (Fact 0.3 (e)) and Riemann Roch (Fact 0.3 (c)) to the vector bundles $E \otimes \mathcal{O}(np)$ and $E \otimes \mathcal{O}(np-q)$. Serre Vanishing implies that $h^1(E \otimes \mathcal{O}(np)) = h^1(E \otimes \mathcal{O}(np-q)) = 0$ for sufficiently large n. Then Riemann Roch implies

$$h^0(E \otimes \mathcal{O}(np)) = deg(E \otimes \mathcal{O}(np)) + (1-g)\,\mathrm{rk}(E \otimes \mathcal{O}(np))$$

and

$$h^0(E \otimes \mathcal{O}(np-q)) = deg(E \otimes \mathcal{O}(np-q)) + (1-g)\,\mathrm{rk}(E \otimes \mathcal{O}(np-q))$$

Recall, the rank and degree of a tensor product of vector bundles E_1 and E_2 with r_i and d_i to denote rank and degree is as follows:

$$\mathrm{rk}(E_1 \otimes E_2) = r_1 \cdot r_2$$
$$deg(E_1 \otimes E_2) = r_1 deg(E_1) + r_2 deg(E_2).$$

We apply these facts to our equalities above to obtain the following relationship.

We obtain:

$$h^0(E \otimes \mathcal{O}(np-q)) = h^0(E \otimes \mathcal{O}(np)) - \mathrm{rk}(E).$$

Recall, we have assumed $rank(E) \geq 2$.

(iii) Let $S(q) = H^0(E \otimes \mathcal{O}(np-q)) \subset H^0(E \otimes \mathcal{O}(np))$.

The vector space $S(q)$ denotes those sections of $E \otimes \mathcal{O}(np)$ which vanish at q and have possible poles of order as large as n at p. Recall, for a line bundle $L(D)$ are meromorphic functions f on X such that $div(f) + D \geq 0$, i.e. functions which have at worst a pole at D (or when $D = np - q$ have at worst poles at p of order n and vanish at q). This vector space $S(q)$ may vary as q varies in C. Consider the union of all such vector spaces:

$$\cup_{q \in C} S(q).$$

As the above argument showed, for each $q \in C - p$, $S(q)$ has dimension: $dim(S(q)) = h^0(E \otimes \mathcal{O}(np)) - \mathrm{rk}(E)$. As we vary q along the one dimensional variety C, the union will be no larger than $h^0(E \otimes \mathcal{O}(np)) - \mathrm{rk} + 1$ (that is, one more dimension larger). Hence

$$dim(\cup_{q \in C} S(q)) \leq h^0(E \otimes \mathcal{O}(np)) - \mathrm{rk}\, E + 1 < h^0(E \otimes \mathcal{O}(np)),$$

where we obtain this last strict inequality whenever $\mathrm{rk}(E) > 1$.

These dimension comparisons give that the dimension of $h^0(E\otimes\mathcal{O}(np))$ is larger than the dimension of the space of all sections of $E\otimes\mathcal{O}(np)$ which vanish at some point $q\in C$.

This allows us to conclude that there is some global section $\sigma\in H^0(E\otimes\mathcal{O}(np))$ such that σ does not vanish on any point of C.

2. Using the nonzero global section σ of the vector bundle $E\otimes\mathcal{O}(np)$ we define a line sub bundle of E with quotient bundle with rank one less than the rank of E.
   (i) First we obtain a map from the trivial line bundle into $E\otimes\mathcal{O}(np)$,
   $$\mathcal{O}\to E\otimes\mathcal{O}(np),$$
   which is defined by sending $1\mapsto\sigma$. Since σ is a global section this defines a morphism on all of $\mathcal{O}$. Furthermore, restriction onto each stalk $\mathcal{O}_p\to E\otimes\mathcal{O}(np)_p$ is injective since $\sigma(p)\neq 0$. Thus, this map gives an embedding.

   (ii) With this map, we now tensor with $\mathcal{O}(-np)$ to obtain a sub bundle of E. This gives the following embedding:
   $$\mathcal{O}(-np)\to E.$$

   (iii) Restricting this map to our vector bundles on the open set $U=C-p$ we obtain the following maps of vector bundles over U (since $\mathcal{O}(-np)|_U\cong\mathcal{O}_C|_U$ because regular functions on U are those which may have at worst poles at p, these are precisely those sections of $\mathcal{O}(-np)$):
   $$\mathcal{O}_C|_U\cong\mathcal{O}(-np)|_U\to E|_U.$$
   This gives the following short exact sequence, when taking the quotient.
   $$0\to\mathcal{O}_C|_U\cong\mathcal{O}(-np)|_U\to E|_U\to E|_U/\mathcal{O}_C|_U\to 0.$$

   (iv) The above is a short exact sequence of vector bundles over the affine variety $U=C-p$. Short exact sequence over affine varieties split. Hence we obtain the following isomorphism:
   $$E|_U\cong\mathcal{O}|_U\oplus(E|_U/\mathcal{O}|_U).$$

3. We can now repeat this process with the vector bundle $E|_U/\mathcal{O}|_U$ (which has rank $\mathrm{rk}(E)-1$ and also has trivial determinat). This will give us a decomposition of $E|_U$ into a direct sum of $\mathrm{rk}(E)-1$ trivial line bundles $\mathcal{O}|U$ with a possible non trivial bundle,
   $$E_U\cong\mathcal{O}|U\oplus\ldots\oplus\mathcal{O}|U\oplus\mathcal{L}|_U.$$

However, since E has trivial determinant, we have that $1 = det(E|_U) = det(\mathcal{L}|_{\mathcal{U}})$ (determinants of tensor products multiply). The only line bundle over U (or any space whatsoever) with determinant one is trivial. Hence, indeed we have

$$E_U \cong \mathcal{O}|U \oplus ... \oplus \mathcal{O}|U.$$

□

[Bea] Beauville, *Conformal blocks, fusion rules and the Verline formula*.
[Beab] ———, *Vector bundles on curves and generalized theta functions: recent results and open problems*.
[BaL94] Beauville and Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419.