NOTES ON AFFINE LIE ALGEBRAS FOR THE CONSTRUCTION OF CONFORMAL BLOCKS VECTOR BUNDLES

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1. Introduction

Our main object of interest is the moduli space $\overline{M}_{g,n}$, the moduli space of stable genus g curves with n marked points and maps from them into other projective varieties. It is thus natural to study vector bundles over these spaces, as the determinant line bundle of a vector bundle give a map from $\overline{M}_{g,n}$ into the projectivization of their global section.

The vector bundles which play the lead character in our story are the vector bundles of conformal blocks. These vector bundles have many amazing properties, some of which I will describe in this talk. They are globally generated and so the maps they give are morphisms. Furthermore, the dual of a fiber over an interior point is isomorphic to the spaces known as "generalized theta functions" which are of independent interest in the study of moduli spaces of vector bundles.

History and Background Isomorphisms were first shown in some special cases by [Ber93], [Tha94], and [Zag95]. The isomorphism in general was shown in [Bea94], [Dri95], [Fal94], and [Kum94].

2. Background on representations of simple finite dimensional Lie algebras

We first go through some background material on simple Lie algebras (Note: semi simple means direct sum of simples, simple means no nontrivial ideals) and their representations.

In this document we refer to $\mathfrak{g}$ as a simple complex Lie algebra. Our primary examples will be $\mathfrak{sl}_{r+1}$ and $\mathfrak{sp}(4)$. Recall briefly that a (complex) Lie algebra, $\mathfrak{g}$, is a vector space over $\mathbb{C}$ equipped with the operation of "bracket" operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, symmetric, and satisfies the *Jacobi identity*. The Jacobi identity means that for any $x, y, z \in \mathfrak{g}$ we have $[x,[y,z]] - [y,[x,z]] + [z,[x,y]] = 0$.

Example 2.1. *The Lie algebra $\mathfrak{gl}_r(\mathbb{C}$ is the collection of $r \times r$ matrices with elements in $\mathbb{C}$ is a Lie algebra with bracket being the commutator. That is $[A,B] = AB - BA$ for any $A, B \in \mathfrak{gl}$.*

Example 2.2. *The Lie algebra $\mathfrak{sl}_{r+1}$ is the collection of matrices in $\mathfrak{gl}_{r+1}$ with trace zero. A typical Cartan subalgebra consists of all those matrices which are diagonal.*

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Example 2.3. *The Lie algebra $\mathfrak{sp}(2r)$ is the collection of matrices, X in $\mathfrak{gl}_{2r}$ such that $SX = -X^tS$, where S is the $2rr$ matrix described as follows*

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

This is often called the symplectic Lie algebra

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (a maximal abelian sub algebra of $\mathfrak{g}$ consisting of semi simple elements). (A semi simple element $h \in \mathfrak{g}$ means the linear map $ad(h) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable). We denote by Φ the root system of $\mathfrak{g}$ with this choice of Cartan subalgebra $\mathfrak{h}$, Δ the base of Φ, and Φ^+ the positive roots.

For any $\mathfrak{g}$ module, V and $\lambda \in \mathfrak{h}^*$, we denote by $V_\lambda = \{v \in V_\lambda : h.v = \lambda(h)v \text{ for all } h \in \mathfrak{h}^*\}$. The *root space* $\alpha \in \Phi$ are the elements $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \alpha(h)x = [h, x], \text{ for all} h \in \mathfrak{h}\}$, that is, the root space is the weight space of the root α with the $\mathfrak{g}$ given the adjoint action. We can think of such weight spaces as the eigen vectors of $\mathfrak{h}$. We denote θ the longest root, that is $\theta \sum_{\alpha_i \in \Delta} \alpha_i$. See Example 2.4 below.

Since commuting linear maps can be simulatiosly diagonalized, $\mathfrak{h}$ acts diagonalizably on $\mathfrak{g}$ with the adjoint representation. Hence, we have a decomposition of $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$ where $\mathfrak{g}^+ = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ and $\mathfrak{g}^- = \oplus_{\alpha>0}\mathfrak{g}_{-\alpha}$.

The Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined as:

$$K(X, Y) = trace(\mathrm{ad}X o \mathrm{ad}Y)$$

is non degenerate on $\mathfrak{h}$ and so the map $\theta : \mathfrak{h} \to \mathfrak{h}^*$ (defined below) is an isomorphism:

$$H \mapsto \theta_H,$$

where

$$\theta_H : \mathfrak{h} \to \mathbb{C}, \text{ such that } \theta_H(M) = K(H, M).$$

For $\alpha \in \mathfrak{h}^*$, define $T_\alpha = \theta^{-1}(\alpha) \in \mathfrak{h}$. The Killing form, along with the isomorphism θ, defines an inner product on $\mathfrak{h}^*$ as follows:

$$< \alpha, \beta >= K(T_\alpha, T_\beta).$$

We often normalize this inner product by using $\check{\alpha} = \frac{2\alpha}{<\alpha,\alpha>}$ on the longer root, so that $< \alpha, \alpha >=< \alpha, \check{\alpha} >= 2$. We call the element,

$$H_\alpha = \frac{2T_\alpha}{< \alpha, \alpha >},$$

the *coroot* of α. In this setup we have, $\lambda(H_\alpha) =< \lambda, \check{\alpha} >$ or $< \lambda, \alpha >$ with the scaled form. The coroot defined above, can also be defined as the unique $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$.

Furthermore, the Lie algebra $\mathfrak{g}$ can be decomposed into $\mathfrak{sl}_2(\alpha)$-triples where $\alpha \in \Phi^+$. An $\mathfrak{sl}_2(\alpha)$-triple is a collection of elements $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_\alpha$, and $h \in \mathfrak{h}$ such that the following relations hold,

$$[h, e] = 2e, [h, f] = -2f, [e, f] = -h.$$

We now say a few words about the weight lattice $P \subset \mathfrak{h}^*$. This lattice consist of linear forms $\lambda \in \mathfrak{h}$ such that $< \lambda, \alpha > \in \mathbb{Z}$ for all $\alpha \in \Phi$. Such a weight is *dominant* if in fact all $< \lambda, \alpha > \geq 0$. Denote by P_+ the set of all such dominant weights of $\mathfrak{g}$. Let $\{H_{\alpha_1}, ..., H_{\alpha_n}\}$ be the set of coroots for $\alpha_i \in \Delta$ in the base (these form a basis for $\mathfrak{h}$. The elements $\omega_j \in \mathfrak{h}^*$ such that $\omega_j(H_{\alpha_i}) = \delta_{ij}$ are a basis of $\mathfrak{h}^*$. We call this dual basis to the coroots of simple roots the *basis of fundamental dominant weights*. All dominant weights are positive linear combinations of these weights. Let $P^+_\ell = \{\lambda \in P^+ :< \lambda, \theta > \leq \ell\}$ where θ is the longest root. We call this set the collection of *dominant integral weights of level ℓ*. We will see the importance of these weights in the discussion of representations of affine Lie algebras.

Example 2.4. *The following is a basis for $\mathfrak{sl}_{n+1}$:*

$$\{E_{i,j} : 1 \leq i \leq n+1, 1 \leq j \leq n+1, i \neq j\} \cup$$
$$\{E_{i,i} - E_{1+i,1+i} : 1 \leq i \leq n\},$$

where $E_{i,j}$ is the elementary $(n+1) \times (n+1)$ matrix with a one in the i row and j column and all other entries zero. A base, Δ, of the root system is given

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \leq i \leq n\}.$$

The highest root is $\theta = \sum_{i=1}^n \alpha_i = \epsilon_1 - \epsilon_{n+1}$.
The $\mathfrak{sl}_2(\epsilon_i - \epsilon_j)$ triples can be described by:

$$\{E_{ij}, E_{ji}, E_{ii} - E_{jj}\}.$$

We will see later the importance of the $\mathfrak{sl}_2(\theta)$-triple: $X_\theta = E_{1,n+1}, X_{-\theta} = E_{1,n+1}, H_\theta = E_{1,1} - E_{n+1,n+1}$.

The "dominant weights" P^+ are given by $\sum_{j=1}^n a_j \epsilon_j$ such that $a_1 \geq a_2 \geq ... \geq$. All dominant weights can be written as a nonzero integer linear combination of the following weights, $\{\omega_i = \sum_{j=1}^i \epsilon_j, 1 \leq i \leq n\}$. We call these the "fundamental" dominant weights. Recall also the relation $\sum_{j=1}^{n+1} \epsilon_j = 0\}$.

Example 2.5. *The following is a base and basis of fundamental dominant weights for $\mathfrak{sp}_4$:*

$$\Delta = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = 2\epsilon\}$$
$$\textit{Fundamental dominant weights}\{\omega_1 = \epsilon_1, \omega_2 = \epsilon 1 + \epsilon_2\}$$
$$\theta = 2\omega_1 = 2\alpha_1 + \alpha_2.$$

2.1. **Irreducible representations of $\mathfrak{g}$.** For every dominant weight $\lambda \in P_+$ there is a unique (up to isomorphism) finite dimensional irreducible $\mathfrak{g}$-module with heights weight λ, we will denote it $V(\lambda)$. We show (in the following construction) that $V(\lambda)$ is generated by a highest weight vector v_λ over $U(\mathfrak{g}_-)$ (where $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ is the triangulation decomposition of $\mathfrak{g}$). We describe the construction to justify this claim.

An irreducible highest weight $\hat{\mathfrak{g}}$-module can be constructed as the largest quotient of the $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U(B)} \mathbb{C}_\lambda$ (where $\mathbb{C}_\lambda$ is the one dimensional representation of $\mathfrak{g}$). It can be shown that such a module is cyclic (see [Hum72] section 20.3). By this property, it must have a maximal proper ideal. We obtain $V(\lambda)$ via this quotient.

The irreducible representation $V(\lambda)$ contains a highest weight vector v_λ with weight λ that is, $x.(v_\lambda) = 0$ if $x \in \mathfrak{g}_\alpha$ with $\alpha > 0$ (i.e., is a highest weight vector) and $h.v_\lambda = \lambda(h)v$ for $h \in \mathfrak{h}$ (i.e., has weight λ). Such an argument shows that the map $\lambda \mapsto [V(\lambda)]$ (where $[V(\lambda)]$ is the isomorphism class of $V(\lambda)$) is bijective.

3. Affine Lie algebras

We now give some background on affine Lie algebras to the extent which are necessary in the construction. For more details on the following information I will refer you to Chapter 7, 9, 10 of [Kac94]. In this talk, we will focus on the affine Lie algebra $\hat{\mathfrak{g}}$ associated $\mathfrak{g}$ defined as follows:

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}((z))) \oplus \mathbb{C}c,$$

where $\mathbb{C}((z))$ is the ring of Laurent polynomials in the variable z (polynomials in z allowed to have infinite negative or positive powers of z). This affine Lie algebra is often called a *central extension* of $(\mathfrak{g} \otimes \mathbb{C}((z)))$ by c or the derived Lie algebra of the loop algebra considered in [?Kac] (definition in chapter 7). We can summarize a few facts about this Lie algebra.

The bracket of two elements in $(\mathfrak{g} \otimes \mathbb{C}((z)))$ is given by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + c \dot{<} X, Y > Res_{z=0}(gdf).$$

It can be checked, due to the residue formula that this bracket defines a Lie algebra (see notes from email June 4th). The center element c is trivial on bracket (since it is in the center).

Similar to the finite case with $\mathfrak{g}$, $\hat{\mathfrak{g}}$ has a decomposition into subspaces (and in fact Lie subalgebras due to the definition of $[,]$ and using residue theorem) $\hat{\mathfrak{g}}_+ = (\mathfrak{g} \otimes z\mathbb{C}[[z]])$ and $\hat{\mathfrak{g}}_- = (\mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}])$ as follows:

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+.$$

This decomposition will play an important role in the remaining discussion of the construction of our conformal blocks. We will denote $\mathfrak{p} := \mathfrak{g} \oplus \mathbb{C}c \oplus \mathfrak{g}_+$ (this is analogous to the "Borel" in the finite case).

In [Kac94] (chapter 7) an explicit description of generators of $\hat{\mathfrak{g}}$ is given. If $\{e_1, ..., e_s, f_1, ..., f_s\}$ is a Chevellay basis of $\mathfrak{g}$, then generators for $\hat{\mathfrak{g}}$ is as follows:

$$E_i = e_i \otimes 1, F_i = f_i \otimes 1, \text{ for } i = 1, ..., s \text{ and } F_0 = f_0 \otimes z^{-}1, E_0 = -\omega(f_0) \otimes z.$$

Where we have chosen $f_0 \in \mathfrak{g}_\theta$ where θ is the largest root of $\mathfrak{g}$ and $e_0 = -\omega(f_0)$ the Chevaley involution (for $\mathfrak{sl}_r$ this involution is the negative transpose). An interesting fact is that $\{e_0, e_1, ..., e_s\}$ generate the finite Lie algebra $\mathfrak{g}$ (as e_0 contains an element of $\mathfrak{n}_-$).

Example 3.1. *For the Lie algebra $\mathfrak{sl}_2$ with basis $\{e, f, h\}$ basis for $\hat{\mathfrak{sl}}_2$. Note for $\mathfrak{sl}_{r+1}$ hieghest root is $\epsilon_1 - \epsilon_{r+1}$ and $X_\theta = E_{1,r+1}$ the elementary matrix.*

4. Properties of irreducible representations of the affine Lie algebra $\hat{\mathfrak{g}}$

We now turn to irreducible representation of $\hat{\mathfrak{g}}$. Such representations must have an action by the center $c \in \hat{\mathfrak{g}}$. It can be checked that this action must be multiplication by some integer $\ell \in \mathbb{Z}$. We call a representation where the center acts by $\ell \geq 0$, a representation of level ℓ and refer to ℓ as the level. Let P_ℓ^+ denote the set of dominant weights of $\mathfrak{g}$ such that $\lambda(H_\theta) =< \lambda, \theta >\leq \ell$ where θ is the longest root of Φ.

We will explore and prove some of the following properties and statements of an irreducible $\hat{\mathfrak{g}}$-module.

Proposition 4.1. *For each $\lambda \in P_\ell$, there exists a unique, irreducible (integrable) $\hat{\mathfrak{g}}$-module (called the integrable highest weight $\hat{\mathfrak{g}}$-module) denoted $\mathcal{H}_\lambda$ satisfying:*

1. *$V(\lambda) \cong \{v \in \mathcal{H}_\lambda : u.v = 0 \text{ for all } u \in \hat{\mathfrak{g}}_+\}$ (where $V(\lambda)$ is the irreducible highest weight $\mathfrak{g}$-module introduced in section 2.1.*
2. *The center elements $c \in \hat{\mathfrak{g}}$ acts on $\mathcal{H}_\lambda$ as ℓId.*
3. *$\mathcal{H}_\lambda$ is generated by a height weight vector, v_λ, with $\hat{\mathfrak{g}}_-$ and the only relation*
$$(X_\theta \otimes 1/z)^{\ell-(\theta,\lambda)+1} v_\lambda = 0,$$
where θ is the longest root of $\mathfrak{g}$ and $X_\theta \in \mathfrak{g}_\theta$ (with $\mathfrak{g}_\theta$ the root space of θ and X_θ the element in $\mathfrak{sl}_2(\theta)$ triple with coroot H_θ).

Example 4.2. *For $\mathfrak{sl}_{r+1}$ we have the following highest root and $\mathfrak{sl}_2(\theta)$-triple:*
$$\theta = \epsilon_1 - \epsilon_{r+1}$$
$$X_\theta = E_{1,n+1}, X_{-\theta} = E_{n+1,1}, H_\theta = E_{1,1} - E_{n+1,n+1}.$$

To explain Proposition 4.1 we first outline the construction of $\mathcal{H}_\lambda$. We begin by defining a few terms and objects and state a few important results from [Kac94] used in the reasoning of the construction of $\mathcal{H}_\lambda$.

We denote by $\mathfrak{g}(A)$ a Kac-Moody algebra associated to an arbitrary $n \times n$ matrix A (see [Kac94] for a more explicit definition). This general object does not play much importance to us, but such a classification of objects contains the affine Lie algebras of our interest. And so the following statements are applicable to us. Let V be a $\mathfrak{g}(A)$-module. We denote $V_\lambda = \{v \in V : h.v = \lambda(h)v, h \in \mathfrak{h}\}$, the λ weight space.

Definition 4.3. *Category $\mathcal{O}$ is the category whose objects are $\mathfrak{g}(A)$ modules V which are $\mathfrak{h}$ diagonalizable with finite dimensional weight spaces such that there exists a finite number $\lambda_1, ..., \lambda_s \in \mathfrak{h}^*$ such that $P(V) \subset \bigcup D(\lambda_i)$, where $P(V) = \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ and $D(V) = \{\mu \in \mathfrak{h}^* : \mu \leq \lambda\}$. We can think of such modules as having finitely many "peaks" or highest weight spaces (this insightful interpretation is thanks to Dr. Boe).*

Definition 4.4. *The Verma module $M(\Lambda)$ is a highest weight $\mathfrak{g}(A)$-module with highest weight Λ that every $\mathfrak{g}(A)$-module with highest weight Λ is a quotient of this module by some sub-module, i.e. looks like $M(\Lambda)/Z$ where Z is some submodule of $M(\Lambda)$. All Verma modules are objects in Category $\mathcal{O}$.*

The following are propositions regarding such modules.

Proposition 4.5. *(See* [Kac94] *Prop 9.2)*

(1) *For all $\lambda \in \mathfrak{h}^*$ there exists a unique Verma module $M(\Lambda)$.*
(2) *As a $U(\mathfrak{n}_-)$-module, $M(\Lambda)$ is a free module of rank one and is generated by a highest weight vector. Here $\mathfrak{n}_-$ is the subalgebra in the triangular decomposition $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (see page 145 of* [Kac94]*).*
(3) *$M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$. As such, the quotient $L(\Lambda) = M(\Lambda)/M'(\Lambda)$ is an irreducible module.*

Proposition 4.6. *(See* [Kac94] *Prop 9.3)) Let V be a nonzero module from category $\mathcal{O}$. Then*

(1) *V contains a nonzero weight vector v such that $\mathfrak{n}^+(v) = 0$.*
(2) *V is irreducible $\Leftrightarrow$ V is a highest weight module and any primitive vector of V is a highest weight vector $\Leftrightarrow$ $V \cong L(\Lambda)$.*
(3) *V is generated by primitive vectors as a $\mathfrak{g}(A)$ module.*

Lemma 4.7. *(See* [Kac94] *Lemma 3.2) We restrict to $\mathfrak{sl}_2(\alpha_i)$-triples. Let V be a $\mathfrak{sl}_2(\alpha_i)$-submodule of a $\hat{\mathfrak{g}}$ module. Denote by e, g, h the elements in $\hat{\mathfrak{g}}$ of the $\mathfrak{sl}_2(\alpha_i)$ triple. Let $c = \alpha_i(h)$ for some $c \in \mathbb{C}$; and let $v \in V$ be such that $h(v) = \alpha_i(h)(v) = cv$. Set $v_j = (j!)^{-1} f^j(v)$. Then: $h(v_j) = (c - 2j)v_j$ and if $e(v) = 0$, then: $e(v_j) = (c - j + 1)v_{j-1}$.*

Lemma 4.8. *(See* [Kac94] *Lemma 10.1) The $\mathfrak{g}(A)$-module $L(\lambda)$ is integrable if and only if $\lambda \in P+$. And particularly we must have $f_i^{<\lambda,\check{\alpha}_i>+1}(v) = 0$ for $i = 1, ..., n$.*

Proof. We prove Lemma 4.8 above.
We can show that this lemma follows from the $\mathfrak{sl}_2$-triple relations in Lemma 4.7 restricted to the e_i, f_i, h_i triples of $\mathfrak{g}(A)$. Since we assume $L(\Lambda)$ is an irreducible highest weight module by 4.6(b) every primitive vector is a highest weight vector. Denote $< \lambda, \check{\alpha}_i >= a_i$. If it were the case that $f_i^{a_i+1}(v)$ was not zero, then from Lemma 4.7 it would follow that $e_i(f_i^{a_i+1}(v)) = (a_i + 1)!(a_i - (a_i + 1) + 1) f^{a_i}(v) = 0$ (follows immediately from Lemma 4.7 if we replace $c = a_i$ and $j = a_i + 1$). And since $[e_j, f_i] = 0$ for all $i \neq j$, the vector $f_i^{a_i+1}(v)$ would be primitive (i.e., killed by all $e_i \in \mathfrak{g}(A)^-$). However, such a vector is not a heights weight vector (as it is not a multiple of v). This contradicts the equivalence in 4.6 (b). We see that vectors of this form are the only such multiple of f which must be zero. □

5. Construction of irreducible $\hat{\mathfrak{g}}$ representations

Now, to construct $\mathcal{H}_\lambda$ we begin with $V(\lambda)$, the irreducible $\mathfrak{g}$-module with highest weight λ such that $< \lambda, \theta > \leq \ell$. We can define an action of $\mathfrak{p} = \mathfrak{g} \oplus \mathbb{C} \cdot c \oplus \hat{\mathfrak{g}}^+$ on $V(\lambda)$ in the following way:

$$c.v_\lambda = \ell v_\lambda \text{ and } \hat{\mathfrak{g}}^+.v_\lambda = 0.$$

We define $\mathcal{V}_\lambda := U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} V(\lambda)$. We can show that such a module is a Verma module (this construction is similar to how the existence of Verma modules was outlined in **??** Chapter 9). This means that $\mathcal{V}_\lambda$ has a unique proper maximal submodule Z_λ.

Furthermore, by the PBW theorem, the universal enveloping algebra (the unitary associative algebra with basis elements given by products and all powers of basis elements of of $\hat{\mathfrak{g}}$) has the following decomposition:

$$U(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}}^+) \oplus U(\mathfrak{h}) \oplus U(\hat{\mathfrak{g}}^-) = U(\mathfrak{p}) \oplus U(\hat{\mathfrak{g}}^-).$$

Hence, we can write $\mathcal{V}_\lambda = U(\hat{\mathfrak{g}}^-) \otimes_{\mathbb{C}} V(\lambda)$ with the extended action of $\mathfrak{p}$ on $V(\lambda)$. By Proposition 4.5 $\mathcal{V}_\lambda$ is generated by a highest weight vector v. And again, we know that there exists a unique maximal proper submodule Z_λ of this module. We denote the irreducible quotient as:

$$H^\ell_\lambda = \mathcal{V}_\lambda / Z_\lambda.$$

We now show this maximal submodule is generated by a vector v_λ over $\hat{\mathfrak{g}}_-$ with the only relation $(X_\theta \otimes z^{-1})^{\ell - \lambda(H_\theta)+1}(v)$.

6. Generators and relations of $\mathcal{H}^\ell_\lambda$

We now explain Proposition 4.1. By Proposition 4.6(b) considering the generators of $\hat{\mathfrak{g}}$ given above, we see that the only potential element of $U(\mathfrak{g}^-)$ which may generate a submodule of $\mathcal{V}_\lambda$ is $F_0(v)$. (Recall the definition of $F_0(v)$ in Section 3, $F_0 = f_0 \otimes z^{-1}$ where $f_0 \in \mathfrak{g}_\theta$). This is because the entire module is generated by v as $U(\mathfrak{g}^-)$-module and all all other terms can 'move' across the tensor. We change our notation slightly to match Beauville's discussion; we write $f_0 = X_\theta$. Now, by Lemma 10.1 and formulas in Lemma 4.7, it follows that the lowest power for which F_0 is primitive is, $< \lambda, \check{\alpha}_0 > +1$. From page 100 of [Kac94], the definition of $\check{\alpha}_0 = \delta - \theta$. This becomes:

$$< \lambda, \delta - \theta > +1 = \ell - < \lambda, \theta > +1.$$

(Note that Kacs' includes $\check{}$ in his notation, however, he not consistent with using this notation for pairing in $\mathfrak{h}$ or $\mathfrak{h}^*$. We try to infer his meaning from context.).

Again, the above shows that $(X_\theta \otimes z^{-1})^{\ell - <\lambda,\theta>+1}(v)$ is the smallest power of $F_0(v)$ which is primitive. By Proposition **??**(c) such an element generates a submodule of $\mathcal{V}_\lambda$ (submodules are only generated by primitive elements) and indeed, such an element generates a maximal submodule, since this again is the *smallest* such power producing a primitive vector. Hence, the unique maximal submodule of $\mathcal{V}_\lambda$ is generated by $(X_\theta \otimes z^{-1})^{\ell - <\lambda,\theta>+1}$. When we take the quotient of $\mathcal{V}_\lambda$ with this submodule, we obtain an irreducible highest weight module (from Proposition 4.5(c)). It then follows that $\mathcal{H}_\lambda$ is generated by the highest weight vector v with the relations that $\mathfrak{g}(v) = 0$ and $(X_\theta \otimes z^{-1})^{\ell - <\lambda,\theta>+1}(v) = 0$.

Remark 6.1. *The level ℓ is relevant to to this discussion in how we defined the action of $U(\mathfrak{p})$ on the irreducible representation V_λ and also how the weight λ pairs with θ. A different level ℓ would specify a different $U(\mathfrak{p})$ action and power.*

This completes our construction of $\mathcal{H}_\lambda$ and gives us specific properties about it with which to work.

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