Seeing Geometry in Algebra: An Introduction to Algebraic Geometry
Davidson College, Algebra Guest Lecture
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"Algebra is but written geometry and geometry is but figured algebra."
- Sophie Germain (1776-1831).

1. Introduction

Algebraic geometers create geometric objects from algebraic structures. In this talk, we will use many ideas that you have learned in your abstract algebra course to construct and *see* geometry in these ideas. These notes are taken from a beautiful introduction text on the subject [1].

2. Defining the Main Object: Affine Algebraic Varieties

The main objects algebraic geometers study are *varieties*. A special kind of variety are *affine algebraic varieties* and these varieties make up the building blocks of most geometric objects studied by algebraic geometers. As we will see, their structure resembles many properties of objects you may have already explored in your algebra class.

Definition 2.1. An *affine algebraic variety* is the common zero set of a collection $\{f_i\}_{i\in I}$ of complex polynomials on complex $n-$space $\mathbb{C}^n$. We write this set the following way,

$$V = \mathbb{V}(\{f_i\}_{i\in I}) \subset \mathbb{C}^n.$$

We use the notation $\mathbb{V}$ because we think of such a set as the *vanishing* set of the polynomials f_i. The I is representing some indexing set. We do not require this set to be finite or have any special properties. It turns out that any variety can be describe as the vanishing set of a finite number of polynomials (this is called *Hilbert basis theorem* and uses properties of Noetherian rings).

We can also define varieties as zero sets of polynomials with coefficients from ANY field! For example, $\mathbb{Z}/p\mathbb{Z}$, a finite field! When we use $\mathbb{C}$ sometimes we call these *complex* algebraic varieties. Some examples are below. In the figures below, the real values where the polynomials vanish are drawn.

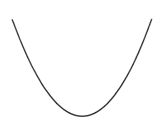

Figure 2.1. $\mathbb{V}(x^2 - y) \subset \mathbb{C}^2$.

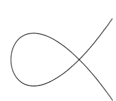

Figure 2.2. $\mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{C}^2$.

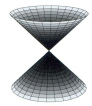

FIGURE 2.3. $\mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{C}^3$.

Example 2.1.
- A parabola! $\mathbb{V}(x^2 - y) \subset \mathbb{C}^2$. Let's draw the real points in this example! See Figure 2.1.
- A curve with a cusp. $\mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{C}^2$. See Figure 2.2.
- A quadratic cone $\mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{C}^3$. See Figure 2.3.
- Here is an example where the variety is the common zero set of two polynomials, $\mathbb{V}(x - y, x^2 - y)$. Points in this variety are zero on each of the polynomials.

Exercise 2.2. Show that the following sets are varieties. To do this, write the collection of polynomials J whose vanishing set describes the set listed.

(1) $\mathbb{C}^n$
(2) $\emptyset$
(3) any point $(a_1, ..., a_n) \in \mathbb{C}^n$

Solution 2.3.

(1) $\mathbb{C}^n = \mathbb{V}(0)$
(2) $\emptyset = \mathbb{V}(1)$ (or we could use ANY constant polynomial).
(3) any point $(a_1, ..., a_n) = \mathbb{V}(\{x_1 - a_1, ..., x_n - a_n\}) \in \mathbb{C}^n$

We have some special names for describing some varieties:
- the *affine plane* $\mathbb{C}^2$,
- a *hypersurface* in $\mathbb{C}^n$ is the zero set of one polynomial. (this generalizes a surface in three space... the dimension is one less than the space it is sitting in).
- a *hyperplane* in $\mathbb{C}^n$ is the zero set of one linear polynomial. (this generalizes a line in the plane). Example: $\mathbb{V}(ax + by + c) \subset \mathbb{C}^2$.

3. Intersections of Affine Algebraic Varieties

Definition 3.1. A *subvariety* W of a variety $V \subset \mathbb{C}^n$ is a variety of $\mathbb{C}^n$ contained in V.

Exercise 3.1.

a) If $\{f_i\} \subset \{g_i\}$ then what is the relationship between the varieties $\mathbb{V}(\{f_i\})$ and $\mathbb{V}(\{g_i\})$? For starters, try to determine the relationship between the varieties $\mathbb{V}(f)$ and $\mathbb{V}(f, g)$ where f and g are two polynomials.

b) Is the **intersection** of two varieties a variety? If f and g are two complex polynomials with n variables and so $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are two affine algebraic varieties in $\mathbb{C}^n$ can you describe polynomials whose zero set is the set $\mathbb{V}(f) \cap \mathbb{V}(g)$?

c) Is the **union** of two varieties a variety? If f and g are two complex polynomials with n variables and so $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are two affine algebraic varieties in $\mathbb{C}^n$ can you describe polynomials whose zero set is the set $\mathbb{V}(f) \cup \mathbb{V}(g)$?

Solution 3.2.

a) $\mathbb{V}(f,g) \subset \mathbb{V}(f)$ and in general, if $\{f_i\} \subset \{g_i\}$ then $\mathbb{V}(\{f_i\}) \supset \mathbb{V}(\{g_i\})$
b) $\mathbb{V}(f) \cap \mathbb{V}(g) = \mathbb{V}(f,g)$.
c) $\mathbb{V}(f) \cup \mathbb{V}(g)\mathbb{V}(fg)$.

In general, the intersections of any number varieties is also a variety and the union of a finite number of varieties is also a variety! Here are some examples. Figure 3.1 is an intersection of two varieties determined by when *both* of the polynomials $x^2 - y$ and $x^3 - z$ vanish. This variety is called the *twisted cubic*. Figure 3.2 is the union of the x-axis and the yz-plane in three space.

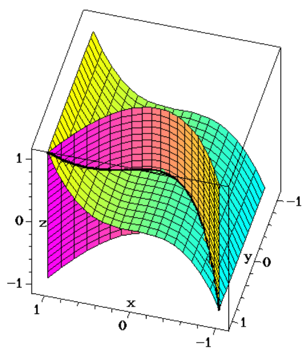

FIGURE 3.1. $\mathbb{V}(x^2 - y, x^3 - z) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x^3 - z) \subset \mathbb{C}^3$.

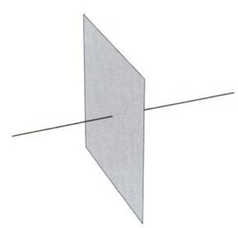

Figure 1.6. $V = \mathbb{V}(y,z) \cup \mathbb{V}(x) = \mathbb{V}(xy,xz)$

FIGURE 3.2. $\mathbb{V}(xy, xz) = \mathbb{V}(y,z) \cup \mathbb{V}(x) \subset \mathbb{C}^3$.

The complements of affine algebraic varieties form a topology on $\mathbb{C}^n$ different than the standard Euclidean topology. This is called the *Zariski-topology* on $\mathbb{C}^n$. The open sets are the complements of varieties and the closed sets are varieties. This is a coarser topology than the Euclidean topology. This means that there are more open sets in Euclidean topology than Zariski-topology. Indeed, the complements of all affine varieties are open in the Euclidean topology, however not all open sets in the Euclidean topology are varieties (they cannot be described as the complements of the vanishing of polynomials).

The requirements on opens sets to form a topology on a space are the following:

(1) finite intersection of open is open
(2) arbitrary union of open is open
(3) $\mathbb{C}^n$ is open

(4) is open

These are all true for complements of varieties (as you showed in some of the above exercises)! So complements of affine algebraic varieties form a topology on $\mathbb{C}^n$.

4. Now, let's do some algebra!

Geometric questions (such as intersections of varieties) can be translated into algebraic problems. To do this, we use a correspondence between geometric objects (varieties) and algebraic objects (ideals). We are able to understand varieties then by a geometric structure. You will see that you know *a lot* about varieties already because you know a lot of algebra from your algebra course this semester!

We haven't been very descriptive yet when explaining the set of complex polynomials whose zero sets define a variety. We will now look more at such sets and see they are quite familiar to you already. In fact, they are ideals! We now use a bit of the ideas you have learned in algebra to describe the elements defining affine algebraic varieties.

Definition 4.1. The space $\mathbb{C}[x_1, ..., x_n]$ is the *complex polynomial ring generated by the variables* $x_1, ..., x_n$. The elements in this ring are polynomials with variables $x_1, ..., x_n$ and complex coefficients.

Example 4.1. For $n = 2$, we can notate the two variables in our polynomials as x and y. Then the ring of complex polynomials in these two variables is $\mathbb{C}[x, y]$. For example, elements in this ring include, $f(x, y) = x^2 - y$, $g(x, y) = x + 2y$, $h(x, y) = 2$ are elements in this ring.

Why is $\mathbb{C}[x_1, ..., x_n]$ a ring?!?!

Exercise 4.2.

a) Show that the set $\mathbb{C}[x_1, ..., x_n]$ forms a commutative ring with the operations of multiplication and addition.
b) What is the additive identity?
c) What is the multiplicative identity?
d) Show that $\mathbb{C}$ is a subring of $\mathbb{C}[x_1, ..., x_n]$.

Solution 4.3.

- $f \cdot g \in \mathbb{C}[x_1, ..., x_n]$.
- $f \cdot 1 = f$.
- $f + g \in \mathbb{C}[x_1, ..., x_n]$.
- $f + 0 = f$.
- The constant polynomials are $\mathbb{C} \subset \mathbb{C}[x_1, ..., x_n]$. They form a ring and are contained in the set of polynomials.

We now want to review a few properties of ideals you have seen in algebra which will be useful to our further study.

Definition 4.2.

- A proper ideal $\mathfrak{m} \subset R$ is *maximal* if the only ideal strictly containing it is R.
- An ideal $\mathfrak{p} \subset R$ is *prime* if for any $fg \in \mathfrak{p}$ we have that either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

- An ideal $I \subset R$ is *radial* if for any $f^n \in I$ we have $f \in I$. For any ideal $I \subset R$, we define the radial ideal of I to be the ideal $\sqrt{I} = \{f : f^n \in I\}$. Hence, I is a radial ideal if and only if $I = \sqrt{I}$.

5. Back to varieties... now with ideals

In this section we are going to show that there is a one-to-one correspondence between varieties V in $\mathbb{C}^n$ and radial ideas $I \subset \mathbb{C}[x_1, ..., x_n]$. This will help illustrate the link between algebra and geometry which demonstrates the power and amazement of algebraic geometry!

Definition 5.1. For a variety $V \subset \mathbb{C}^n$. Define the following set,

$$\mathbb{I}(V) = f \in \mathbb{C}[x_1, ..., x_n] : f(x) = 0 \text{ for all } x \in V\}.$$

This is called the vanishing ideal of V.

Exercise 5.1.

a) Let $V \subset \mathbb{C}^n$ be any affine algebraic variety. Justify the name of the set $\mathbb{I}(V)$. That is, show that for any variety $V \subset \mathbb{C}^n$ the set $\mathbb{I}(V) \subset \mathbb{C}[x_1, ..., x_n]$ is an ideal.
b) Furthermore, show that $\mathbb{I}(V)$ is a prime and radical ideal of $\mathbb{C}[x_1, ..., x_n]$.
c) If $W \subset V$ are two varieties, then what is the relationship between the radial ideals $\mathbb{I}(W)$ and $\mathbb{I}(V)$?

Solution 5.2.

- Must show $\mathbb{I}(V)$ is a subring and closed under multiplication by elements in $\mathbb{C}[x_1, ..., x_n]$. If f and g both vanish on V then clearly so does there sum and product so $\mathbb{I}(V)$ is a subring of $\mathbb{C}[x_1, ..., x_n]$. And if $k \in \mathbb{C}[x_1, ..., x_n]$ is any polynomial, then $k \cdot f$ also vanishes on V.

- If $W \subset V$, then $\mathbb{I}(W) \supset \mathbb{I}(V)$

Exercise 5.3.

a) Show that $V \subset \mathbb{V}(\mathbb{I}(V))$.

b) Show $\mathbb{V}(\mathbb{I}(V)) \subset V$ to conclude that $\mathbb{V}(\mathbb{I}(V)) = V$. To start, let $V = \mathbb{V}(\{f_i\})$ be defined by the vanishing of the polynomials $\{f_i\}$. Use that $\{f_i\} \subset \mathbb{I}(V)$.

c) Conclude that $V = \mathbb{V}(\mathbb{I}(V))$.

Does this map have an inverse? Consider composing, $\mathbb{V}(\mathbb{I}(V))$ and $\mathbb{I}(\mathbb{V}(I))$, our maps have inverses if each of these compositions returns the input. Check in the next exercise that this second composition may fail.

Exercise 5.4. Is it true that $\mathbb{I}(\mathbb{V}(I)) = I$ also for any prime ideal $I \subset \mathbb{C}^n[x_1, ..., x_n]$? Give an example of two different primes ideals in $\mathbb{C}[x_1, ..., x_n]$.

Solution 5.5. Examples, consider the ideas $\{x^2 - y\}$ and $\{(x^2 - y)^2\}$. Are these ideals prime? Are these different or the same? What are the varieties these ideas give? Both give the same variety that we previously saw in earlier example.

This example makes us see the importance of radical ideals in this correspondence. It is a big theorem (Theorem 5.1) due to Hilbert which allows us to conclude that the the correspondence $\mathbb{I}(\mathbb{V}(I)) = I$ for *radical* ideals. We do not give the proof here. Try to show it using what you know from your algebra class!

Theorem 5.1. *(Hilbert's Nullstellensatz). For any ideal $I \subset \mathbb{C}[x_1, ..., x_n]$,*

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

So if I is already radical, then $\mathbb{I}(\mathbb{V}(I)) = I$.

This means there is a one-to-one correspondence between affine algebraic varieties in $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[x_1, ..., x_n]$! So we can either work with a radical ideal or a variety!!!

$$\{\text{affine algebraic varieties } V \subset \mathbb{C}^n\} \leftrightarrow \{\text{radical ideals } I \subset \mathbb{C}[x_1, ..., x_n]\}.$$

6. Concluding exercises

To conclude, here are some fun exercise you can do with varieties using what you already know about algebra.

Exercise 6.1. Can you describe the vanishing ideal of a point $(a_1, ..., a_n) \in \mathbb{C}^n$? That is, describe the ideal $\mathbb{I}((a_1, ..., a_n))$. Can you show that this ideal is a maximal ideal in $\mathbb{C}[x_1, ..., x_n]$?

From your work in some of the previous exercises, how does the correspondence between varieties and radical ideals work with inclusions when translating? Use the following exercise to help describe inclusions in this translation.

Exercise 6.2.

a) If V and W are two varieties such that $W \subset V$, then what is the relationship between $\mathbb{I}(W)$ and $\mathbb{I}(V)$.
b) If $J \subset I$ are two radical, prime ideals such that $J \subset I$, then what is the relationship between $\mathbb{I}(J)$ and $\mathbb{I}(I)$?

References

[1] Karen Smith, *An Invitation to Algebraic Geometry*, Universitext (1979), Springer, 2000. ↑